RAMSEY THEORY ON THE INTEGERS: ADDITIVE VERSUS MULTIPLICATIVE

MIQUEL ORTEGA

1. Ramsey Theory

Ramsey theory is a branch of combinatorics that, in very broad terms, is concerned with finding a given well-organised substructure in structures of large size. This kind of statement appears all over mathematics, with examples in geometry, number theory, logic, or combinatorics. The prototypical question of the theory, phrased in an informal manner, is the following. Suppose there is a group of n people, which may or may not know each other. If n is large enough, can one always find three people that either all know each other or are all strangers? Ramsey's theorem gives an affirmative answer to this question.

A natural way to state Ramsey's theorem is in the language of graph theory. Consider a graph of n vertices, where each vertex represents one of the members of the group, and place a red edge between two vertices if they know each other and a blue edge otherwise. The question, then, is whether a red-blue colouring of the edges of K_n must contain a monochromatic triangle. In fact, Ramsey's theorem is slightly more general, since it guarantees the existence of cliques of any one given size.

Theorem (Ramsey's theorem)**.** *Given $k \in \mathbb{N}$, there exists an n such that any 2-colouring of the edges of K_n contains a monochromatic K_k.*

The *Ramsey number of order k*, frequently denoted by $R(k,k)$, is the minimal n such that the theorem holds. For example, Figure 1 shows that $R(3,3) > 5$, since it provides a 2-colouring of K_5 with no monochromatic K_3, and it is a nice exercise to check that, in fact, it holds that $R(3,3) = 6$. The next Ramsey number, $R(4,4)$, is known to be 18, but it exact value of $R(5,5)$ is already unknown, and obtaining precise estimates for $R(k,k)$ for large k is a notoriously hard problem.

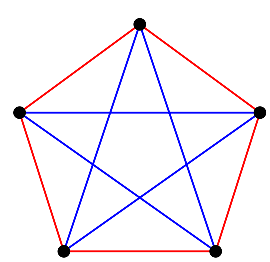


Figure 1. Colouring of K_5 with no monochromatic K_3

At first, the theorem might come across as an interesting but slightly shallow puzzle. However, it has spawned deep and interesting mathematics, because it concisely captures the tension between how regular a deterministic graph can be and how much structure a random one contains. In fact, Ramsey theory has many connections with the theory of random graphs, as the following examples show:

- Erdős [**Erdos1947**] proved in 1947 an asymptotic lower bound on the size of the Ramsey numbers of the form
$$R(k,k) \geq (1+o(1))\frac{k}{\sqrt{2}e}2^{k/2}$$
as k grows. In fact, this lower bound is achieved by taking a random 2-colouring of the edges in K_n, which, as it turns out, contains no monochromatic cliques of size k with positive probability.

  The proof of this bound is usually recognised as the first explicit appearance of the probabilistic method in combinatorics, where a deterministic combinatorial problem is approached by framing it in a probabilistic manner. Since then, the method has been applied in countless problems, giving birth to the whole field of probabilistic combinatorics.
- One of the first improvements on the upper bounds for Ramsey numbers, proved by Thomason [**Thomason1988**] in 1988, was achieved using the notion of pseudorandomness, which measures how much a deterministic set *appears* to be random in the sense that certain statistics are close to those one would expect from a random set. This notion has turned out to be very fruitful, with applications in many different problems in combinatorics.
- Perhaps the most direct connection with random graph theory, and the one we are more interested in for this note, is the study of Ramsey properties in random graphs. To fix ideas, consider the Erdős-Rényi random graph model $G(n,p)$, where every edge of K_n is sampled independently with probability p. One of the main topics of random graph theory consists in studying the threshold of a given property, that is, how large must p be so that a certain property holds in $G(n,p)$ with high probability. In the case of Ramsey properties, this problem was studied in a series of papers in the nineties by Rödl and Ruciński [**Rodl1995**], where they proved, among other things, that when $p \gg n^{-1/(k-1)}$, with high probability, any 2-colouring of $G(n,p)$ contains a monochromatic K_k. Since then, the threshold for many more properties has also been studied and the proof of their result revisited with different techniques.

2. Ramsey theory over the integers

So far, we have seen examples of Ramsey properties over graphs and random graphs. Another possible setting in which to study Ramsey properties is over the integers. In this case, the well-organised substructure one looks for is usually the solution of some equation with variables taking value in $[n] = \{1, \ldots, n\}$ as n grows large. One of the first examples of this approach is Schur's theorem, which says the following.

Theorem (Schur's theorem)**.** *For every number of colours r, there exists n such that any r-colouring of the set $[n]$ contains a monochromatic solution to*

$$x + y = z. \tag{1}$$

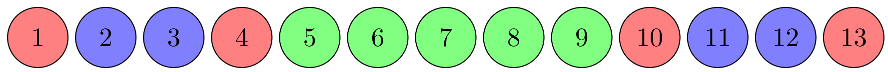


FIGURE 2. Schur colouring of $\{1, \ldots, 13\}$

Such a solution is called a Schur triple*.*

The Schur number $S(r)$ is the largest n such that the previous theorem does not hold. In other words, it is the largest value of n such that $[n]$ admits a *Schur colouring*, an r-colouring with no monochromatic solutions to (1). For example, in Figure 2 we see a Schur colouring of $\{1, \ldots, 13\}$ using three colours. This shows that $S(3) \geq 13$, and it turns out there exists no such colouring of $\{1, \ldots, 14\}$, so that $S(3) = 13$.

Versions of Schur's theorem have also been studied over random sets of integers. For example, one could be interested in determining when the random set $[n]_p$, where every element of $[n]$ is sampled independently with probability p, admits a t-colouring with no monochromatic Schur triples. Graham, Rödl and Ruciński [**Graham1996-fw**] proved the following.

Theorem (Schur's theorem for random sets)**.** *The following holds with high probability.*

- *If $p \gg n^{-1/2}$, any 2-colouring of $[n]_p$ contains at least $\Omega(n^2p^3)$ monochromatic Schur triples.*
- *If $p \ll n^{-1/2}$, there exists a 2-colouring of $[n]_p$ with no monochromatic Schur triples.*

This kind of theorem tells us that, if p is large enough, and as far as the Schur property is concerned, the random set $[n]_p$ behaves similarly to the full set of integers $[n]$. While studying Schur properties of random sets of integers, Graham posed the following question, which belongs to the deterministic setting, but could be helpful for better understanding the problem in both settings. What is the minimum number of monochromatic Schur triples in a 2-colouring of $[n]$? In other words, he asked for a quantitative version of Schur's theorem, giving the minimum amount of monochromatic solutions. This was answered few years later by Robertson and Zeilberger [**Robertson1998-sy**] and Schoen [**Schoen1999-bz**], who proved the following result.

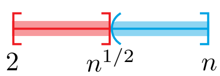
Theorem. *Any 2-colouring of $[n]$ contains at least $n^2/22 + O(n)$ monochromatic Schur triples, and there exists a colouring achieving this bound.*

This result is interesting for at least two reasons. In the first place, it is useful to understand better the behaviour of Schur colourings that avoid (or, rather, minimize) monochromatic Schur triples, since they might provide insights into the nature of such colourings. In the second place, because it is an example of a property that is important in Ramsey theory, where the optimal colouring minimizing the copies of the substructure is not the random colouring. This indicates that understanding the random colouring is not always enough to understand the Ramsey property.

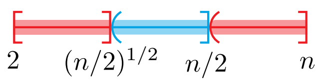
3. What about products?

In the 2022 British Combinatorial Conference, Prendiville asked whether there is a version of Theorem 2 which holds for a multiplicative version of the problem. More concretely, he asked what is the minimum number of monochromatic solutions to $xy = z$ in an r-colouring of $\{2, \ldots, n\}$. In joint work with Aragão, Chapman and Souza, partially developed during a visit to IMPA of the author under the Randnet program, we studied precisely this question.

When working with multiplication, the scale of the numbers involved plays a much more important role. For example, any three numbers of size greater than $\sqrt{n}$ will never produce a product Schur triple, since the product of any two of them is larger than n. In the case of two colours, this suggests a colouring where the numbers from 2 up to roughly $n^{1/2}$ are coloured red and the rest are coloured blue, which has $\sqrt{n}\log(n)$ monochromatic solutions (many less than in the additive case!). This colouring may be represented graphically as



Is this the best possible colouring? Well, one can slightly adjust it by setting the last $n/2$ numbers to red, which may be depicted as follows.



This colouring gives a better constant, with $\frac{n^{1/2}\log n}{2\sqrt{2}}$ monochromatic solutions up to lower order terms. In [**Aragao2024**] we proved that this is actually the least possible number of monochromatic solutions.

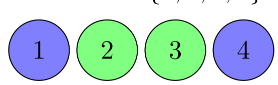
Theorem. *Any 2-colouring of $[2, n]$ will contain at least*

$$\frac{1}{2\sqrt{2}} n^{1/2} \log n$$

monochromatic solutions to $xy = z$.

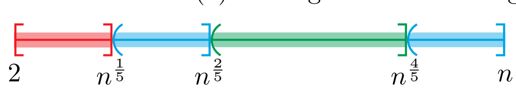
We also explored the situation for more colours. It turns out there is a nice connection with the usual additive Schur numbers $S(t)$. For example, consider the following Schur colouring of the numbers $\{1, 2, 3, 4\}$.

(2)



This may be transformed into a 3-colouring of $[n]$ with few monochromatic solutions to $xy = z$ by painting numbers according to their scale. Concretely, one may paint the first $n^{1/5}$ a different colour, say red, and then paint the numbers between $n^{i/5}$ and $n^{(i+1)/5}$ with the colour of i in (2). This gives the following colouring of $[2, n]$.

(3)



It is not too hard to see that because there are no solutions to $x + y = z$ or $x + y + 1 = z$ in (2), the only monochromatic solutions to $xy = z$ in (3) use colour red. This gives $n^{1/5} \log n$ monochromatic solutions, up to multiplying by a constant, and it turns out that this is the best possible 3-colouring if we do not care about such constants. In fact, this same idea can be generalized as follows to an arbitrary amount of colours.

Theorem 3.1. *For every number of colours $r \geq 2$, we have*

- *Any r-colouring of $[2, n]$ must contain at least $\Omega(n^{1/S(r-1)} \log n)$ monochromatic solutions to $xy = z$.*
- *There exists an r-colouring of $[2, n]$ with less than $O(n^{1/I(r-1)} \log n)$ monochromatic solutions to $xy = z$.*

Here, by $I(r)$ we mean the infimum of all real number $T \in \mathbb{R}$ such that every r-colouring of the closed real interval $[1, T]$ has a monochromatic solution to $x+y = z$.

The number $I(r)$ is a relaxed notion of the Schur number, where one colours the whole real interval instead of just integer numbers. It is straightforward to see that $I(r) \leq S(r)$, and it is also true, although less obvious, that $I(r + 1) \geq 3S(r) - 1$. Therefore, the two numbers cannot be too far apart, and, in fact, they are equal for $r \leq 3$, while it is unknown if they are equal or not for larger values of r.

In conclusion, we were able to partially answer our original question regarding monochromatic solutions to $xy = z$ in r-colourings of $[2, n]$. Concretely, we fully answered it for $r = 2$, we established the correct order of the answer for $r \leq 4$, and provided the first known non-trivial bounds for larger values of r. In doing so, we discovered a connection between the usual additive version of Schur numbers and the multiplicative analogue of the Schur equation. Hopefully, this connection goes further than our result and provides new insights into the nature of Schur numbers and arithmetic Ramsey theory.